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1.—THEORY OF LIQUID FLOW THROUGH CONES

(An approximate General Solution for Conical Tubes of Small Angle).

By W. N. BOND, M.A., D.Sc., F.Inst.P., Lecturer in Physics, University of Reading.

ABSTRACT.

This Paper gives an approximate general solution of the hydro-dynamical equations for liquid flow through conical tubes of circular section, the errors due to the approximation being small for converging cones of small angle and for flow through similar diverging cones till near the speed at which the theory predicts turbulent motion. The theory is compared with experimental results previously published.

INTRODUCTION.

IT is notorious that the general hydrodynamical equations are of such a form that in few cases (if indeed in any) has a general solution of them been obtained covering the whole range between the stage where the pressure-gradient is almost independent of the acceleration of the fluid and the stage where it is almost entirely dependent on the acceleration. In these circumstances, an approximation may be worth consideration. The case which is solved approximately in this Paper consists of the steady flow of liquids through small-angled conical tubes of circular section. The approximation is such that the magnitude of the errors introduced can be estimated with the help of the approximate solution obtained.

THE TWO APPROXIMATIONS.

Choosing rectangular axes, with that of x coinciding with the axis of the cone, we have the three usual hydrodynamical equations such as

$$\frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \dots \dots \dots (1)$$

where p denotes the pressure, u, v, w are components of the velocity at a point, μ and ρ are the viscosity and density of the fluid. (The hydrostatic pressure gradients have for convenience been considered zero.)

Let us first assume that the angle of the cone is small. Denoting the radius of the transverse section by r_m , the distance from the (imaginary) apex of the cone by R and the volume of fluid passing through the cone per second by Q , we can find the order of magnitude of the terms involved. Thus, u is of order Q/r_m^2 , v and w of order Q/Rr_m . Each differentiation with respect to x produces one higher power of R in the denominator, and each with respect to y and z introduces one higher power of r_m in the denominator. Omitting, then, the second order terms in equations (1), we obtain

$$\frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho \frac{Du}{Dt} \cdot \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \dots \dots \dots (2)$$

This result shows that for cones of small angle the transverse pressure gradients may be neglected, and the only way in which the equations differ from those for

parallel stream-lines is in the term Du/Dt .—This result might have been anticipated. The closeness of this first approximation depends only on how small the angle of the cone may be. In the extreme cases of very fast and very slow flow, the errors introduced for a cone of semi-angle 10° are about 1 per cent. and 4 per cent. respectively.*

If the terms in ρ may be omitted from the general equation (1), the flow is in straight stream-lines that when produced would intersect at the apex of the cone.† (This case, that of very slow flow, can be realised to a very close approximation indeed in practice.) If, on the other hand, the terms in μ may be omitted from the general equations (1), the flow would again be in straight stream-lines that when produced would intersect at the apex of the cone. (This case can only be roughly approximated to in practice. Close to the walls of the tube the terms in μ always have an appreciable effect in the case of a converging cone; and in the case of a diverging cone the flow becomes turbulent.) The velocity variation over the transverse section is in these two cases respectively approximately parabolic and approximately constant except near the walls. In the case of a long converging cone, these cases would occur very far from and very near to the apex respectively; and it is evident that the stream-lines in the intervening parts of the tube must be curved. However, as the *second approximation*, for any short part of the cone considered, let us assume that the flow is in straight stream-lines that pass when produced through the apex of the cone. This assumption will introduce a small error in the term Du/Dt (see below), and this error remains even for indefinitely small angles of cone.

With the second approximation, equations (2) reduce to

$$\frac{dp}{dx} = 2\frac{\mu}{r} \cdot \frac{du}{dr} + 2\frac{\alpha\rho}{r_m} u^2 \quad \dots \dots \dots (3)$$

where the cone has a semi-angle α (α positive if the tube diverges in the direction of motion); and there is a pressure-gradient dp/dx (positive when the pressure increases in the direction of motion), at a place where the transverse section of the tube has a radius r_m .

By proceeding with the calculation using the approximate equation (3) (see next section of the Paper), the velocity distribution over the transverse section can be deduced, and knowing this at different sections of the cone, the slightly curved stream-lines may be plotted. If this is done, it is found that for a converging cone, in the worst case ($|\alpha| \rho Q / r_m \mu = 12$ and $r = 0.4 r_m$ approximately), the inclination of the stream-line differs by about 11 per cent. from that assumed in the approximation (and is, therefore, still about 1 per cent. in error). As the assumed inclinations are correct at $r=0$ and $r=r_m$, and as the approximation only affects one of the terms in equation (3), the maximum error in the estimates of u and Q will probably be not more than 5 per cent. (For larger and smaller values of $|\alpha| \rho Q / r_m \mu$ the error will decrease steadily to zero.) For diverging cones the error will only become appreciable as the stage of instability is neared, and as the prediction of instability will not be affected by the removal of the approximation, and is affected by the nature of the entrance to the cone, the error is not of much importance.

* Bond, Proc. Phys. Soc., Vol. 34, p. 188 (August, 1922).

† Harrison, Proc. Camb. Phil. Soc., Vol. 19, pp. 311-12; and Bond, Phil. Mag., Vol. 50, p. 1,059 (1925).

THE APPROXIMATE GENERAL SOLUTION.

In the solution of equation (3) three cases arise :—

Case I.— α and dp/dx both negative. *Converging Cones.*

If the velocity at the walls be assumed zero we have

$$u = \sqrt{\frac{r_m}{2\alpha\rho} \cdot \frac{dp}{dx}} \cdot \tanh \left\{ \frac{1}{2\mu} \sqrt{\frac{\alpha\rho}{2r_m} \cdot \frac{dp}{dx}} \cdot (r_m^2 - r^2) \right\} \dots \dots \dots (4)$$

This equation may further be integrated to give the volume of liquid passing through the tube per second, Q ,

$$Q = -\frac{2\pi\mu r_m}{\alpha\rho} \cdot \log \cosh \left\{ \frac{r_m^2}{2\mu} \sqrt{\frac{\alpha\rho}{2r_m} \cdot \frac{dp}{dx}} \right\} \dots \dots \dots (5)$$

Case II.— α positive and dp/dx negative. *Slow flow through Diverging Cones.*

Again assuming zero velocity at the walls of the tube, we obtain

$$u = \sqrt{-\frac{r_m}{2\alpha\rho} \cdot \frac{dp}{dx}} \cdot \tan \left\{ \frac{1}{2\mu} \sqrt{-\frac{\alpha\rho}{2r_m} \cdot \frac{dp}{dx}} \cdot (r_m^2 - r^2) \right\} \dots \dots \dots (6)$$

$$Q = \frac{2\pi\mu r_m}{\alpha\rho} \cdot \log \sec \left\{ \frac{r_m^2}{2\mu} \sqrt{-\frac{\alpha\rho}{2r_m} \cdot \frac{dp}{dx}} \right\} \dots \dots \dots (7)$$

According to these equations the velocity should become infinite at the axis of the tube, and the rate of flow become infinite at a pressure-gradient given by

$$\frac{dp}{dx} = -\frac{2\pi^2\mu^2}{\alpha r_m^3 \rho}$$

The flow would then, presumably, become turbulent.

Case III.— α and dp/dx both positive. *Fast flow through Diverging Cones.*

The solution of equation (3) corresponding to an increase of pressure along a diverging cone (such as is observed experimentally) takes the form

$$u = \sqrt{\frac{r_m}{2\alpha\rho} \cdot \frac{dp}{dx}} \cdot \tanh \left\{ \frac{1}{2\mu} \sqrt{\frac{\alpha\rho}{2r_m} \cdot \frac{dp}{dx}} \cdot r^2 + \text{constant} \right\}$$

If the velocity of the liquid at the walls be assumed zero, this equation gives negative values of u , the whole becoming the convergent case (equation 4) already considered. Hence (unless slip be assumed to occur at the walls, and the very peculiar solution yielded by the above equation accepted as feasible), we must conclude that stream-line motion is impossible under the conditions of Case III.

DISCUSSION OF THE EQUATIONS.

The solution obtained for convergent flow, and the absence of a solution for stream-line flow in the diverging case except at low speeds, are in general agreement with experiment. The fact that turbulence may sometimes occur in converging cones* is not inconsistent with this theory.

In discussing the solutions of equation (3) it is convenient to note that equations

* Gibson, Proc. Roy. Soc., A., 83, p. 376 (1910).

(5) and (7) represent relationships between two non-dimensional functions, such as $(\alpha \rho Q)/(r_m \mu)$ and $(r_m^4 \cdot dp/dx)/(\mu Q)$; and that the form of the variation of velocity with distance from the axis, expressed by equations (4) and (6), depends on only one non-dimensional function, which may be either of the above or any product of powers of them.

In Fig. 1, equations (5) and (7) are used to plot a relationship between the above non-dimensional functions. (For convenience, the sign of α has here been called positive.) This diagram may be compared with similar diagrams of experimental results given in an earlier Paper,* showing the relationship between Q and h/Q (h being the pressure-difference in centimetres of water column, between the liquid at two parts of the tube).

For slow motion, equations (4) and (6) represent the velocity varying para-

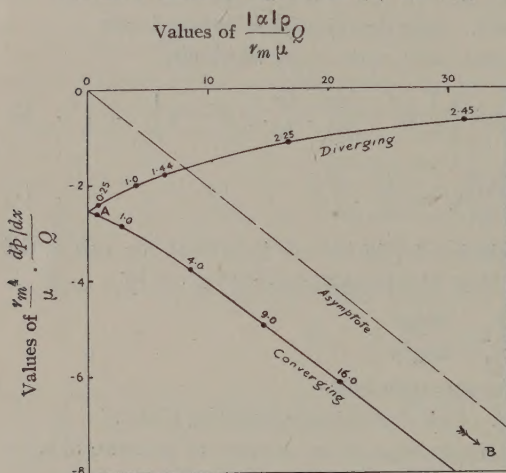


FIG. 1.

bolically over the transverse section; and equations (5) and (7) reduce to Poiseuille's equation. This is in agreement with experiment, and is represented by the point *A* in Fig. 1. For convergent flow at large rates, equation (4) denotes a velocity which is almost constant over the transverse section, except close to the walls of the tube. This is in agreement with experiment, as is also the limiting rate of flow predicted, equation (5) reducing to Bernoulli's expression for frictionless flow. This state corresponds to the limit *B* in Fig. 1.

For moderately slow flow, when the kinetic energy of the

liquid is just starting to be of importance, the slope of the lines at *A* in Fig. 1 can be shown to be equal to two-thirds of the final slope of the lower line at *B*. The experimental results† gave the slope at *A* roughly equal to that at *B*. (When those results were published the slope at *A* had been expected to be about double that at *B*.)

Fig. 2 illustrates for converging flow the velocities and the velocity distribution across the transverse section for various pressure-gradients (the numbers by the curves being proportional to the pressure-gradient). The corresponding points on Fig. 1 are indicated.

For diverging flow, corresponding curves are shown in Fig. 3, the outermost curve representing the limiting case. When the very large velocities predicted for points near the axis are attained, a large rate of flow corresponds to a small pressure-gradient, and the curve in Fig. 1 asymptotes to the axis. In the experiment

* Bond, loc. cit., Figs. 3, 4 and 5.

† Bond, loc. cit., p. 194.

previously referred to, such a state was found, except that the pressure seemed to rise slightly in the direction of motion, whereas according to the theory it would be expected to fall slightly.

At yet higher speeds turbulence would be expected to set in near the axis of the tube, the liquid moving in a narrow jet near the centre of the diverging cone. This effect is well known, and was suggested as an explanation in the author's Paper previously mentioned. This turbulence should occur first at the narrow entrance to the cone, and, as the direction of the walls changes at this point, the theory cannot be supposed to apply accurately and predict the exact critical pressure-gradient.

When the turbulent state is reached, the theory described cannot yield numerical results. However, if the liquid be supposed to move very fast and turbulently in a small and roughly parallel-sided jet from the entrance down the centre of the

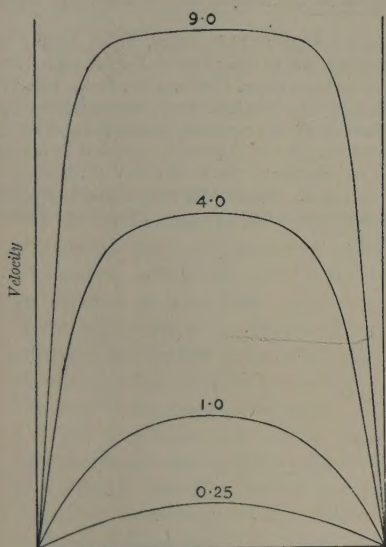


FIG. 2.—LONGITUDINAL SECTION OF TUBE.

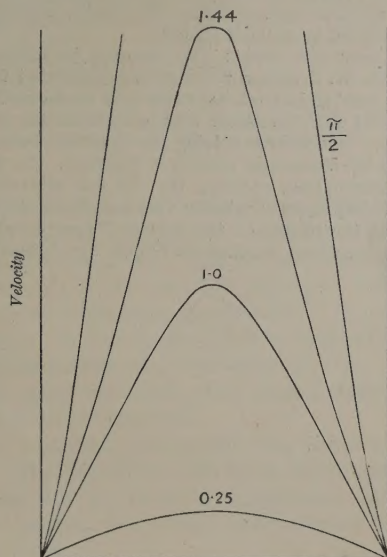


FIG. 3.—LONGITUDINAL SECTION OF TUBE.

diverging cone, and this jet at some point beyond spreads out at an angle larger than that of the cone, till it fills the whole cone, the increase in pressure with distance may, over this diverging part of the jet, be larger than for a jet of angle equal to the cone. If, as the rate of flow is increased, the point of divergence moves nearer to the entrance of the cone, a large upward pressure-gradient would be observed when the point of divergence reached the part of the tube at which the pressure-gradient was being measured. This would correspond to the humps in Figs. 4 and 5 in the Paper previously referred to. Finally, when the point of divergence had almost reached the entrance to the cone, the liquid in the part of the tube experimented on would move turbulently in a cone nearly filling the tube, and the upward pressure-gradient might be expected to be about that deduced for frictionless flow, as was found to be the case in the experiments quoted.

DISCUSSION.

Dr. A. FERGUSON : The Society is indebted to Dr. Bond for the solution which he has given of this very interesting problem. It would add much to the interest of his Paper if he would compare his solution with one given by Professor Gibson in the *Philosophical Magazine* about 1909. Professor Gibson, if I remember rightly, puts

$$v=w=\frac{\partial p}{\partial y}=\frac{\partial p}{\partial z}=0$$

so that the fundamental equations reduce to

$$-\frac{\partial p}{\partial x} = -\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho u \cdot \frac{\partial u}{\partial x}$$

which, transferred to polar co-ordinates x, θ , integrates quite readily. This solution was not discussed in such detail as Dr. Bond's, and a comparison of the simplifying assumptions made would not be without interest.

AUTHOR'S reply : The equation in Professor Gibson's Paper (*Phil. Mag.*, July, 1909), to which Dr. Ferguson refers, differs somewhat from equation (2) of the present Paper, and I am also doubtful about the validity of its derivation (*see* Harrison, *Proc. Camb. Phil. Soc.*, Vol. 19, pp. 311-12). Professor Gibson assumes the stream-lines to be straight lines through the cone apex ; and later in solving the equation seems to assume that the velocity varies parabolically over the transverse section of the cone. In the present Paper the former is assumed as a first approximation, enabling the (curved) stream-lines to be plotted to the next order of accuracy ; and the parabolic velocity variation is definitely not assumed. These differences in assumption make the results of the present Paper considerably different from those of Professor Gibson, some of whose conclusions I think are definitely in error.

II.—THE STRUCTURE OF THE SMOKE PARTICLES FROM A CADMIUM ARC.

By H. P. WALMSLEY, *M.Sc.*

Received April 21, 1927.

(Communicated by Dr. A. B. WOOD.)

ABSTRACT.

Using the powder method of X-ray analysis, the particles dispersed in air from a cadmium arc are found to be isometric crystals of cadmium oxide. From X-ray data, a density of 8.16 was obtained for the primary particles in the smokes—the normal density of cadmium oxide. Photometric measurements of the breadth of the lines showed that the primary crystals were of colloidal dimensions, values of 5.8×10^{-6} cms. and 4.9×10^{-6} cms. being obtained in two cases. On aggregation, the ultramicroscopic crystals are shown to grow along binary axes of symmetry—i.e., they tend to unite on their 110 faces. Photomicrographs of the deposits obtained from cadmium oxide smokes are shown, together with models of aggregates built on these lines. The models give many of the characteristic structures which are observed in the complex particles. Previous measurements on the smoke particles are discussed in the light of these results.

I.

BY means of the powder method of X-ray analysis, an attempt has been made to determine the density and average size of the primary particles dispersed as a cloud, by arcing between cadmium electrodes in air. The arc was enclosed in a box from which the cloud could be withdrawn, by means of an aspirator, through an intense electric field, where the cloud particles were precipitated. A sample of the dark brown precipitate when examined by the Debye-Scherrer method using copper K α radiation gave a photograph characteristic of a crystalline powder. The position and intensities of the lines were in good agreement with those recorded by Scherrer⁽¹⁾ for cadmium oxide. This identified the material.

To determine the lattice dimensions, a second photograph was taken of the powder mixed with a little nickel oxide.⁽²⁾ The lines due to the latter were employed to calibrate the film and avoid corrections due to errors of adjustment, such as imperfect centering of the specimen in the camera. The results were reduced by the method of least squares and are tabulated below.

Cadmium Oxide.

$$\text{Cu K}\alpha_1 = 1.537\text{\AA}, \text{ Cu K}\alpha_2 = 1.541\text{\AA}, \text{ NiO } \alpha = 4.705\text{\AA}.$$

No.	<i>hkl</i>	θ	$\sin \theta / \sqrt{\Sigma h^2}$	No.	<i>hkl</i>	θ	$\sin \theta / \sqrt{\Sigma h^2}$
4	113	65.44°	0.1630 ₀	9	224	106.39°	0.1634 ₅
5	222	68.79°	0.1634 ₅	10 a_1	{ 115 333 }	116.78°	0.1639 ₀
6	004	81.55°	0.1632 ₆	a_2		117.31°	0.1639 ₃
7	133	90.80°	0.1633 ₅	11 a_1	044	136.03°	0.1639 ₀
8	024	93.88°	0.1633 ₈	a_2	...	136.76°	0.1639 ₁
				Mean ...		0.16355	

$$\text{The lattice constant } a = \frac{\lambda}{2} \cdot \frac{\sqrt{\Sigma h^2}}{\sin \theta} = \frac{1.537\text{\AA}}{2 \times 0.16355} = 4.699 \pm 0.002\text{\AA}.$$

Assuming four molecules to the unit cell, the density

$$\rho = 4 \times \frac{128.4 \times 1.649 \times 10^{-24}}{(4.699 \times 10^8)^3} = 8.164 \pm 0.004$$

According to Scherrer, the crystal structure of cadmium oxide is similar to that of rock salt with a value of $a = 4.72 \text{ \AA}$. Its mean density according to Landolt Börnstein's tables is 8.15.

It therefore appears that the molecules of cadmium volatilized by the arc, after oxidation, reassemble with the symmetry of the same isometric space lattice, as is found in laboratory specimens of cadmium oxide. Since there is nothing to hinder the growth of the particles in the presence of the supersaturated vapour, the internal structure will be bounded by facets. The smoke particles are therefore tiny crystals.

The size of the particles was determined from the equation

$$B = 2 \sqrt{\frac{\log_e 2}{\pi}} \cdot \frac{\lambda}{D} \cdot \frac{1}{\cos \theta} + b,$$

where D is the depth of the layers contributing to the reflection at the glancing angle θ . To determine B (Halbwertsbreite), photometric measurements of opacity were taken across the lines on the photographic film, excluding those occurring at very small angles and very large angles on account of their curvature. The coefficient of $1/\cos \theta$ is obtained from the slope of the straight line obtained by plotting the values of B as ordinates and those of $1/\cos \theta$ as abscissæ. This gave a value of $D = 5.8 \times 10^{-6}$ cms.

The homogeneous particles of the cloud—the primary particles—are, therefore, ultramicroscopic crystals, precisely similar in density and internal structure to macroscopic crystals of cadmium oxide.

II.

After dispersion, the primary particles in smoke clouds form secondary or complex particles by aggregation. With Cd O there is a marked tendency to produce chain-like structures. In sparse clouds the structure is occasionally seen in the ultramicroscope, but it may be observed under the microscope (with suitable illumination) in deposits collected on a glass plate by sedimentation (Fig. 3). With dense smokes, particles diffuse and adhere to the roof of the smoke box and to projections on its walls, and grow as fine threads under the action of gravity. In an electric field these tend to lie along the lines of force, and the chains grow as arborescent structures from each electrode. The growth may continue until they stretch completely across the field. If a horizontal sheet of paper is just pierced from below by the lips of charged wires, the threads as they deposit under gravity mark out the lines of force of the electric field, just as iron filings do in the analogous magnetic fields.

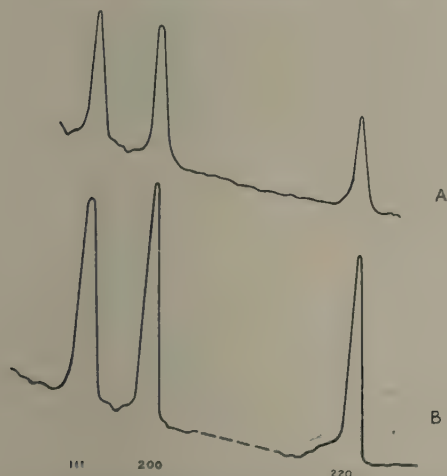
As the structures are aggregations of crystals, it may be expected that the space lattices of adjacent particles will exhibit the usual tendency to set themselves parallel on linking up—the phenomenon of parallel grouping. This occurs frequently with isometric crystals; the crystallisation of native copper and gold are usually quoted as examples. On this hypothesis, the long chains or threads will

form along the direction of an axis of symmetry, and if we employ such a thread instead of a cylinder of powder in the Debye-Scherrer method, it should be possible to identify the axis. The restricted orientation of the particles changes the intensities of the lines on the film relative to those obtained with irregularly distributed particles for only planes whose normals are perpendicular to the axis of the camera are able to reflect X-rays. Compared with the case where the same particles are distributed irregularly, the mathematical expressions for the intensities of a given reflection will differ only in the factor which represents the number n of possible reflecting faces. For example, if a tetragonal axis lies parallel to the axis of the camera, n for the 100 and 110 faces is 4, and for the 111 faces is zero. When the particles are irregularly oriented, the corresponding values are 6, 12 and 8. Thus, the ratios of the intensities are as $4/6 : 4/12 : 0$ —i.e., as $2 : 1 : 0$. Table II gives the relative changes in the intensities of the reflections from the principal planes for each of the three types of axes along which aggregation may occur.

TABLE II.

Plane.	Vertical axis in camera.		
	Tetragonal.	Trigonal.	Digonal.
100	2	0	2
110	1	3	1
111	0	0	3

Some of the fine thread-like structures formed from a dense cloud were attached



A Chain-like structures placed along axis of camera.
B Precipitated material placed along axis of camera.

FIG. 1.

lengthwise along a hair and a powder photograph taken. The film was measured photometrically and compared with one of similar density taken with irregularly distributed particles. The results for the three principal reflections are shown graphically in Fig. 1. The intensity of reflection is proportional to the area of the curve above the general background. With the oriented particles, it is clear that the 111 reflection is enhanced relatively to the 200, and the 220 is weakened. The direction of growth of the chain-like structures is therefore the direction of a binary axis of symmetry.

Let I^n be the intensity of the reflections when the particles are oriented at random and I_0 that due to the thread-like structures on

the hair. In Table III are given the ratios I_n/I_0 , which would have been observed if the 220 reflection had been equal in each case. The fourth column gives the

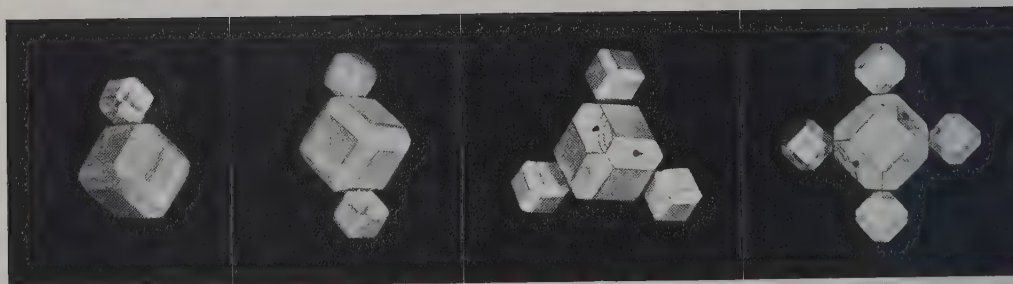
theoretical values of the ratio on the assumption that 20 per cent. of the particles are lying with a binary axis of symmetry parallel to the axis of the camera, and 80 per cent. are irregularly distributed. The agreement of these values with the observed seems tolerably good when it is remembered that, owing to uncertainty in assigning values for the intensity of the background of the film at each line, accurate measurements of the intensities of the weaker ones are difficult to obtain.

TABLE III.

hkl	I_n	Ratio of intensities.		hkl	I_n	Ratio of intensities.	
		Found.	Calc.			Found.	Calc.
111	77	1.40	1.40	400	12	1.26	1.20
200	96	1.20	1.20	331	22	1.08	1.00
220	100	1.00	1.00	420	32	1.05	1.00
311	53	1.12	1.00	422	35	1.00	1.00
222	24	1.38	1.40	511	33	1.00	1.10
				333			

The value 20 per cent. is perhaps as high as can be expected. For obvious reasons, it is impossible to pull the chain-like structures taut when attaching them to the hair in the camera, and so get the maximum effect. Further, for each crystal unit of the chain there are five other binary axes, all inclined to the direction of the hair along which growth is possible. From the widths of the lines, the average dimensions D of the primary particles in the chain-like structures was found to be 4.9×10^{-6} cms. Thus, the structures are built up of ultramicroscopic primary crystals united on their rhombic dodecahedral (110) faces. They are skeleton crystals. Similar skeleton forms composed of macroscopic crystals are familiar in crystallography as crystallites⁽³⁾ or dendrites. They are forms showing parallel growth. The skeleton structure will not be entirely destroyed when the complex particles are obtained from the cloud in bulk by either sedimentation or electrical precipitation. Thus, although the individual units in the deposit possess the normal density of cadmium oxide, that of the material in bulk will be less.⁽⁴⁾

There is an interesting analogy between the growth of the skeleton forms from the tiny crystals and the growth of the crystal from vapour. Natural crystals of CdO occur rarely. They have been found as regular octahedra.⁽⁵⁾ They have been prepared artificially by Florence⁽⁶⁾ as microscopic octahedra by fuzing CdO with borax. Prepared by burning metallic cadmium in a stream of oxygen, they appear as cubes, and as cubes showing octahedran faces.⁽⁵⁾ The crystals occasionally found in the muffles in zinc smelting processes occur as octahedra⁽⁷⁾; at times modified by 100, 110 and 210 faces. The octahedra and cube faces appear to be the dominant forms. The 110 faces tend to disappear during growth, so occur less frequently, and it is on these faces that the crystals in the smokes usually unite. Thus, when two primary crystalline particles of cadmium oxide aggregate, the colliding crystal particle tends to attach itself, preferentially to precisely the same face as would a colliding molecule during the growth of the crystal from vapour. Repetition of this process naturally produces a skeleton form. From the point of view of the Gibbs-Helmholtz equation, the 110 faces, growing more rapidly than either the 100 or 111 faces, possess the maximum adhesional surface energy.



a

b

c

d

FIG. 2.

III.

The general configuration of the crystalline complex resulting from aggregation of the particles depends upon the conditions under which the building material is supplied. The production of long tree-like forms from the upper walls of the smoke-box and from the electrodes of an electric field through which a flow of smoke is maintained, obviously arises by growth along a concentration gradient. In the absence of such gradients one would expect the skeleton forms to develop more symmetrically; but, once development has commenced, the subsequent growth from primary particles may be considered as before as a growth along a concentration gradient. Complex particles like the one illustrated in Fig. 3*a* clearly arise from the union of two or more skeleton forms. The general appearance of the dendritic forms is therefore governed by the disposition of those particles which united to form the centre of growth.

The primary particles of the clouds are not of uniform size. In fact, the sizes seem to range between comparatively wide limits. As the collision frequency is greatest between particles of extreme sizes,⁽⁸⁾ the most frequently occurring type of binary particle will be one containing a large and a small primary, united on a 110 face. This is represented by the model in Fig. 2*a*, where the cadmium oxide particles are represented as cubes showing 110 faces (in conformity with their shape when produced by burning cadmium in oxygen). The large particle may be considered as the centre from which the dendrite grows by union with the small one. Growth will continue for a time mainly by the addition of the smaller particles. A second small particle uniting with the binary *may* join on to any one of the 22 exposed 110 faces, or may even unite on the vacant part of the 110 face of the large particle which already carries the first small particle. Considerations of size make a union on another 110 face of the large particle more probable, but do not indicate which face. Independent evidence exists that the motions of the particles are not entirely governed by the laws of chance, so that union with any one of the vacant 110 faces of the larger particle is not equally likely. If we may extend the previous analogy between crystal growth from vapour and the dendritic growth of aggregates, it is perhaps not unreasonable to suppose that just as the molecules of a crystal assemble with rhythmic symmetry, a tendency to produce symmetrical arrangements exists with the particles. Crystals frequently show a tendency on association to produce symmetrical groupings. With crystals of low symmetry this often results in the simulation of higher forms (mimetic twinning).

Assuming, therefore, that the small crystal particles attach themselves, if possible, symmetrically about the large one, which is acting as a centre of growth combinations with two, three and four particles will tend to associate as in the models of Fig. 2. These should be commonly occurring basic forms on which the dendritic structures may be conceived to develop by growth along concentration gradients.

The cruciform structure of Fig. 2*d* is found in the complex shown in the photograph of Fig. 3*c*. The spots there are diffraction images formed by oblique illumination of the particles under the ultramicroscope, using the violet line from a mercury vapour arc. The average distance along the axes between the centres of the spots is 740μ . Thus, if the crystal dimensions D were 740μ the structure would be represented by the model of Fig. 2*d*, with five particles equal in size. A

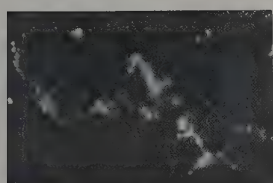
cube edge of 7.4×10^{-5} cms. is a very large but not impossible size for a primary particle. As there is no definite information of the actual sizes of the particles forming the cross, the presence of much smaller particles in the cloud would suggest that the structure is more like the model of Fig. 3e, where the five large particles, now each less than 7.4×10^{-5} cms. edge, are united by smaller ones. If the dimensions of the large particles are but two or three times greater than those of the small ones, the ratio of the amount of light scattered, according to Rayleigh's law, will be 2^6 or 3^6 respectively—i.e., the smaller particles will scatter only 1.56 per cent., or 0.14 per cent. of the amount scattered by the larger ones. This could hardly have been detected; so making the links uniting the large particles invisible.

The three-limbed complex in the photomicrograph of Fig. 3d is more interesting. The sharpness of focus for all the spots shows that the three arms are coplanar, or nearly so. This arrangement is only possible with growth along binary axes of symmetry. The arms show trigonal symmetry about a perpendicular through the centre of growth, which immediately makes growth along a tetragonal axes impossible. The other possible growth axis—a trigonal axis—is equally not responsible for the structure, for in this case, although threefold symmetry is possible, the arms would not be coplanar, but the centre of growth would be supported by the arms in the manner of a tripod above the glass surface carrying the deposit. With the microscope objective used, the central spots would be out of focus, just as are parts of the neighbouring large complexes. The structure arises from the basic model of Fig. 2c, and the crystallographic evidence of this quite common type of complex confirms the X-ray data. Fig. 3f is a model of the complex built on similar lines to that in Fig. 3e. Fig. 3g is a reduced photograph of this model when illuminated obliquely as were the particles in the photomicrographs.

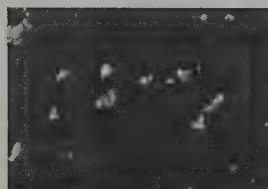
IV.

In a Paper⁽⁹⁾ on the conductivity of cadmium oxide clouds, the author found it convenient to classify the ions present as small ones and large ones. The size of the large ions was found to increase as the clouds aged, and although it was not possible to obtain exact determinations of their mobilities and sizes, limiting values were obtained for the ions in a particular case. Treated as spherical in shape, the upper limit to the radius of the small ions was found to be 3.35×10^{-6} cms. After 20 minutes the lower limit for the radius of the large ions was 2.2×10^{-5} cms., and as the cloud aged some of the particles attained at the upper limit radii of 9.11×10^{-5} cms. and 17.6×10^{-5} cms. The dimensions of the primary crystals measured in this Paper— $D=5.8 \times 10^{-6}$ cms. and $D=4.9 \times 10^{-6}$ cms.—fall within the limits of size previously given for the small ions. Apparently the majority of the small ions are primary crystal particles carrying charges, and the large ones are dendrites or crystal clusters. This is confirmed by recent work of Patterson and Whytlaw Gray,⁽¹⁰⁾ who have obtained estimates of the sizes and densities of individual particles in cadmium oxide smokes, using Millikan's method for oil drops. Determinations gave values for the radius in 10^{-5} cms. of 2.07, 2.42, 5.70, 6.38, 6.60, 7.60, 7.81 and 13.1. These mostly fall within the limits of size of the large ions given above. The mean value obtained for their density, 0.51, is consistent with the dendritic structure of the aggregates.

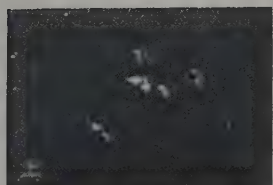
The estimate of size by both methods rests on the assumption that the resistance to the motion of the particles is governed by Stokes' law. Since the resistance



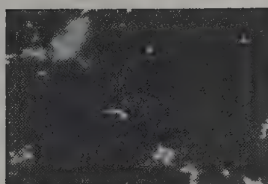
a



b

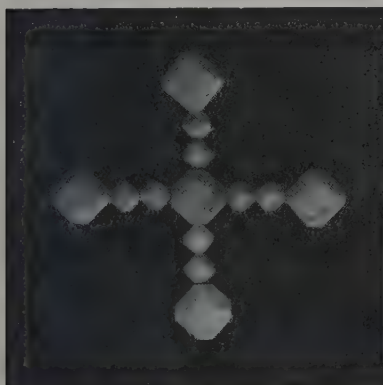


c

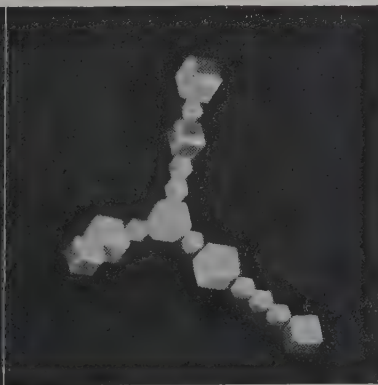


d

0 ————— 50 ————— 100 SCALE OF μ



e



f



g

FIG. 3

To face page 12]

is apparently of the same order when the particles are moving under gravity and when moving in an electric field, it would appear that in general the complex particles in sparse smokes must be fairly symmetrical in shape, and photomicrographs seem to confirm this (Fig. 3*d*). As a corollary, the chain-like structures are perhaps not to be looked upon as a free normal growth. Although they certainly may arise by the union of structures like the models of Fig. 2 (*a* and *b*), a very large number must arise by disruption from the larger complexes, many of which are inherently unstable.⁽¹¹⁾ The disruptive process is quite evident if the growth of dendritic forms is observed in an electric field. It will be noticed that as the structure grows particles and branches occasionally break away from the dendrite growing from one electrode, shoot across the field and attach themselves to the branches of the dendrite growing from the other electrode, and vice versa.

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- (2) Brentano and Dawson, *Phil. Mag.* (7), 3, p. 411 (1927).
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- (4) Kohlschutter and Tüscher, *Zeit. für Electrochem.*, 27, p. 1 (1921).
- (5) Wittich and Neuman, *Centr. Min.*, p. 550 (1901).
- (6) Florence, *Neues Jahrb. Min.*, 2, p. 137 (1898).
- (7) Von Werther, *Jour. für Prak. Chem.*, 55 (1852).
- (8) See succeeding note.
- (9) Walmsley, *Phil. Mag.* (7), 1, p. 1266 (1926).
- (10) Patterson and Whytlaw Gray, *Proc. Roy. Soc., A*, 113, p. 312 (1926).
- (11) Walmsley, *Phil. Mag.* (7), 3, p. 587 (1927).

DISCUSSION.

Dr. G. SHEARER said that the author could have saved himself a great deal of trouble by using a large photographic plate, or a photographic film surrounding the hair, instead of a narrow axial strip. The equation giving the relation between brightness and the size of the particles was primarily applicable only to cubic crystals, and he was surprised that there should be a linear relation between B and $\sec \theta$ in the present case ; the point needed fuller investigation.

The AUTHOR, in reply, said that his chief technical difficulty had been to get the particles arranged on the hair. His results could scarcely be taken as proving a linear relation between B and $\sec \theta$; he had wanted a rough estimate of the size, and had obtained this by finding the mean straight line through the points plotted.

III.--THE POWER-FACTOR AND CAPACITY OF THE ELECTRODES AND BASE OF TRIODE VALVES WITH SPECIAL REFERENCE TO THEIR USE IN THERMIONIC VOLTMETERS.

By G. W. SUTTON, *B.Sc.*

Received May 19, 1927.

ABSTRACT.

The Paper discusses the conditions under which a three-electrode valve-voltmeter should be operated to ensure a minimum power consumption, and at the same time to give indications closely proportional to the square of the input voltage. A simple method of adjusting the operating voltages to fulfil the necessary conditions is described.

THE requirements for a valve-voltmeter to be used for high-frequency resistance measurements may be summarised as follows:—

- (1) That it shall impose the smallest possible load on tuned circuits to which it may be connected.
- (2) That its indications shall be as closely proportional to the square of the applied voltage as possible.
- (3) That it shall be rapid in indication.

(1) An inherent disadvantage of the voltmeter, when compared with the thermojunction, is that it involves connection with tuned circuits at points at high potential as well as at "earthed" points. Thus the effective capacity of the tuned circuit is increased, the magnitude of the induced voltage is likely to be altered, and any losses in the voltmeter are added to those in the tuned circuit. The additional effective capacity of the order of 5 to $10\mu F$ is troublesome at times, but is much less so with the valve-voltmeter than with the electrostatic voltmeter, since in the latter case it varies appreciably with the instrument reading.

The losses in the voltmeter are a much more serious point. In his Paper on a "Direct Reading Thermionic Voltmeter,"* E. B. Moullin shows that the effective resistance of his anode bend type voltmeter is of the order of 0.5 megohm at full-scale reading of about 1.5 volts (R.M.S.), and varies very considerably with the applied voltage. He quotes a case in which the instrument introduces an additional decrement of about 0.015 into a tuned circuit. Since the decrements of good inductance coils range from 0.005 to 0.1, his subsequent statement that "In cases where the decrement introduced by the voltmeter is of importance, it can be determined with sufficient accuracy to enable the necessary allowance to be made," stands in need of experimental demonstration. In any case, it may even double or treble the decrement of the circuit under investigation. By the use of larger values of mean anode and mean negative grid voltage, than those employed in the Moullin instrument, the effect of grid current can be very materially reduced, but it has been found from time to time that the usual form of valve-holder and base may introduce considerable additional loss, as might be expected from their material and construction.

* Journal I.E.E. (December, 1922).

L. Hartshorn and T. I. Jones have investigated the losses associated with valve capacities.* In both Papers the Schering bridge is used, at audio-frequencies. With condensers of such high power-factor as these appear to possess, the influence of a frequency change of, from 1,000 cycles to, perhaps, 1,000 $kC.$, is likely to be very considerable. It seemed well worth while therefore, to make a few further measurements at radio-frequencies. Moreover, Messrs. Hartshorn and Jones have considered the general case in which both anode and grid potentials fluctuate. For the present purpose we need only be concerned with fluctuating grid potential, since as large a condenser as is necessary will invariably be connected from anode to filament. The effective capacity of the voltmeter is therefore, to a very close approximation, the sum of the mutual capacities from grid to filament and from grid to anode.

In one of the above Papers† it is stated that "It would not be safe to make a quantitative deduction of power-factor, but an examination of the results indicates that power-factor is higher in the glass (the pinch) than in the black insulating compound." It was thought that this point needed further investigation, since it indicates the necessity for the use of specially constructed valves, of the "V" type, for instance, for valve-voltmeters.

From the results on which this statement is based it is possible, by assuming the valve elements to constitute a circuit as illustrated in Fig. 1, to estimate the values of the various parallel capacities and resistances, and hence the corresponding power-factors.

Results of measurements on the Schering bridge from L. Hartshorn's Paper :

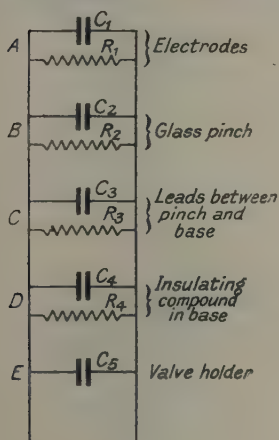


FIG. 1.

		$C \mu F$	P.F.
$A+B+C+D+E$...	3.82	0.21
$B+C+D+E$...	1.65	0.31
$C+D+E$...	1.11	0.24
$D+E$...	0.97	0.23
E	...	0.26	0.00

Assuming a frequency of $6,000/2\pi$ (for convenience in arithmetic; the actual value is not quoted), the following values may be deduced:—

TABLE I.

	C_1 etc. μF	R_1 etc. Megohms	P.F.
A	2.17	578	0.13
B	0.54	676	0.47
C	0.14	3850	0.31
D	0.71	748	0.31
E	0.26	0	0

* L. Hartshorn and T. I. Jones, "The Inter-Electrode Capacities of Thermionic Valves." Exp. Wireless (February, 1925).

† L. Hartshorn, "The Input Impedances of Thermionic Valves at I.F." Proc. Phys. Soc. (February, 1927).

This certainly indicates that the power-factor of the pinch is higher than that of the insulating compound in the base, and that both are exceptionally high. But it also allots a power-factor of 0.13 to the air-insulated electrodes and 0.31 to air-insulated leads; whilst the valve-holder has immeasurably small power-factor. It is true that the valve-holder used was of sound design and mounted on ebonite, but it is surprising to find its insulation resistance so much higher than that of a few centimetres of air-insulated conductors.

Some results obtained previously by the present writer had indicated that the losses to be expected, both in the glass pinch and also in the base, are of a much lower order than those just quoted. Before giving these results, it may be as well to describe the method by which they were obtained.

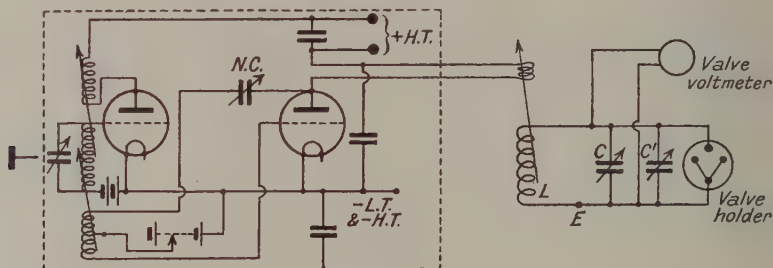


FIG. 2.

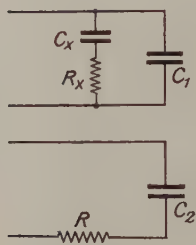


FIG. 2A.

C_x = effective capacity of valve, etc.

C_1 = remaining tuning capacity.

C_2 = capacity equivalent to C_x , C_1 , R_x .

R_x = equivalent series resistance of valve losses.

R = resistance in series with C_2 equivalent to R_x , C_x , C_1 .

Following E. Mallett and A. D. Blumlien,* the shielded generator unit used consisted of an oscillator and an amplifier (or preferably "separator") valve. Increased security against reaction between the circuit LC and the oscillator valve was obtained by balancing

the grid-anode capacity of the separator valve by the neutralising condenser $N.C.$ The circuit LC consisted of a low-resistance coil tuned to resonance by the condensers C and C' ; C' being a small calibrated vernier enabling measurements of small changes of capacity to be made to an accuracy of about $0.02\mu\mu F$. Across these condensers a valve-voltmeter and a well-designed valve-holder were connected. The two filament and the anode sockets of the latter were joined together and to the point "E", (the "earth" side of the circuit) and the grid to the high potential side.

When determining the power-factor of a valve or valve-base, the capacity was measured, first, by substitution, using C' alone for the purpose. The valve was then again placed in the holder (care being taken to see that it was always pressed home in the sockets to the same point), the circuit brought to resonance and the

* Journal I.E.E., "A New Method of H.F. Resistance Measurement" (December, 1924).

voltmeter reading noted. Next the valve was removed, the circuit retuned, and the new voltmeter reading, invariably larger than the first, was taken. Two or three resistances, of values found experimentally to be approximately equivalent to losses in the valve, were then inserted in turn at the point *E*. From these readings the resistance which, if placed in series with all the tuning capacity in the circuit, is exactly equivalent to the losses in the valve, may readily be calculated. Calling this value *R*, and that of the equivalent series resistance of the valve alone *R_x* (see Fig. 2A), we have, equating admittance operators,

$$j\omega C_1 + \frac{1}{R_x - j\frac{1}{\omega C_x}} = \frac{j\omega C_2}{1 + j\omega R C_2}$$

or

$$C_1 + \frac{C_x - j\omega C_x^2 R_x}{1 + R_x^2 \omega^2 C_x^2} = \frac{C_2 - j\omega C_2^2 R}{1 + R^2 \omega^2 C_2^2}$$

In this case

$$R_x^2 \omega^2 C_x^2 \text{ and } R^2 \omega^2 C_2^2 \text{ are } \ll 1$$

So that

$$C_1 + C_x - j\omega R_x C_x^2 = C_2 - j\omega C_2^2 R$$

Also

$$C_1 + C_x = C_2 \quad (\text{very nearly})$$

And hence

$$R_x = \left(\frac{C_2}{C_x} \right)^2 R \text{ to a close approximation.}$$

Thus, from *R* and the known capacities, *R_x*, and hence the power-factor of the valve, may be calculated.

In Table II the results of a number of tests carried out from time to time by this method are summarised.

TABLE II.

$(f=650 \text{ kC.})$				$C(\mu\text{F.})$	P.F.
(a) Modern "low-loss" valve holder (sockets mounted on narrow insulating ring of large diameter)				1.4 ₃	0.04
(b) {	French D.E. valve	7.0 ₀	0.02 ₁
(c) {	Ditto base alone	1.1 ₀	0.1 ₀
(d) {	Valve base found previously to have very low insulation resistance	2.4 ₃	0.37
(e) {	English D.E. valve	4.9 ₈	0.03 ₈
	Ditto base alone	1.5 ₈	0.05 ₅
	Ditto remainder alone—i.e., pinch, electrodes and leads	3.0 ₈	0.02 ₀
	$(f=545 \text{ kC.})$				
(f) {	"R" valve. Metal ring on cap	5.0 ₃	0.06 ₃
	Ditto base alone	1.8 ₃	0.17
	Ditto remainder alone	2.9 ₀	0.00 ₃
(g) {	"R" valve. Another manufacturer	4.6 ₂	0.04 ₉
	Ditto base alone	1.7 ₂	0.09 ₈
	Ditto remainder alone	2.5 ₈	0.02 ₈ *
(h) {	D.E.R. valve. Bakelite cap	4.4 ₁	0.01 ₅
	Ditto base alone	0.7 ₅	0.05 ₄
	Ditto remainder alone	3.2 ₃	0.00 ₁₅
(i) {	D.E.R. Old pattern, before the use of "gettering"	5.1 ₁	0.03 ₈
	Ditto base alone	1.6 ₁	0.05 ₄
	Ditto remainder alone	3.2 ₃	0.02 ₇

* It was noticed that the under surface of the glass pinch was oily. This may have accounted for the comparatively high value of the power-factor in this case as compared with the others.

From this it appears that, at least at radio-frequencies, the insulation in the base of the average receiving valve has a much higher power-factor than the glass pinch, and occasionally it may be very high indeed (*see (d)* in Table II). The power-factor of the pinch and electrodes, on the other hand, is frequently so low as to introduce completely negligible losses into an ordinary tuned circuit. In general, therefore, it should be sufficient merely to remove the cap from an ordinary receiving valve for use as a valve-voltmeter.

As an instance of the effect which neglect of this precaution may be expected to have when measuring H.F. resistance, the following experiment may be quoted. An anode-bend voltmeter was added to a circuit a portion at a time, the circuit consisting of an inductance of $140\mu H$ and a condenser of $400\mu F$.

TABLE III.

(a) Resistance of tuned circuit alone	3.6 ₈ ohms
(b) With valve-holder, galvanometer and batteries connected; but no valve or shunting condenser from anode to filament	3.8 ₃ "
(c) Valve added, but no filament or anode current	4.0 ₂ "
(d) Filament current switched on... ..	4.0 ₁ "
(e) Anode current switched on	4.3 ₅ "
(f) Shunting condenser ($1.5\mu F$) added added	4.0 ₈ "

In view of the above results an ordinary D.E.R. valve was selected, and the base removed by clipping the connecting wires from the pins and soaking the cement with which the cap is held to the glass in hot water for a few hours. After cleaning the surface of the glass around the points where the electrodes emerge, the valve was fitted, in an inverted position, into a small wooden box. The grid lead was kept as short and as far removed from the remaining apparatus as possible. By this means the equivalent resistance of the valve was reduced to that due to the losses in the pinch, and to any grid current which might flow. The effective capacity was found to be $4.5\mu F$; and the power-factor, with no filament current, 0.005.

To limit the effect of grid current as far as possible it was decided to employ a grid bias of -4.5 volts. A few measurements of the increase of series resistance of the circuit used above due to dielectric loss and grid current together were then carried out with decreasing values of negative grid bias, the applied voltage being adjusted in each case so that the positive peak value carried the instantaneous grid potential to -0.5 volts. The results of this test were as follows:—

TABLE IV.

Mean grid potential Increase of series re- sistance of the cir- cuit	-4.2	-4.1	-4.0	-3.9	-3.8	-3.0	-2.6 volts
	0.04 ₃	0.05 ₁	0.06 ₈	0.07 ₇	0.06 ₈	0.09 ₀	0.10 ohms

The decrement of the circuit used in the above experiments is 0.014. Moullin's anode-bend voltmeter would add, on full-scale deflection (using his figure of 0.75 megohm), 0.0025, or about 18 per cent. The voltmeter discussed above would add, on full-scale deflection and using -4.2 volts grid-bias, 0.00024, or about 1.8 per cent. Even with -2.6 volts grid bias it only becomes 0.0005.

Against this improved performance must be set the fact that three circuit adjustments are necessary in place of one. It should be noticed, however, that

the combined adjustment of filament emission and grid bias in the Moullin instrument is only satisfactory if the emission per ampere of filament current may be assumed to be constant. This is certainly not the case with dull-emitting filaments, where the voltage to give constant emission may change as much as from 3.42 to 2.75 volts in 200 hours of use.* This point has been borne in mind in designing the present instrument.

(2) The chief advantage of a square-law valve voltmeter is that it simplifies and accelerates the process of calculation in high-frequency resistance measurements. No adjustments additional to those which would be required in any case to maintain constancy of calibration are necessary, and the use of the somewhat cumbersome calibration graph is avoided.

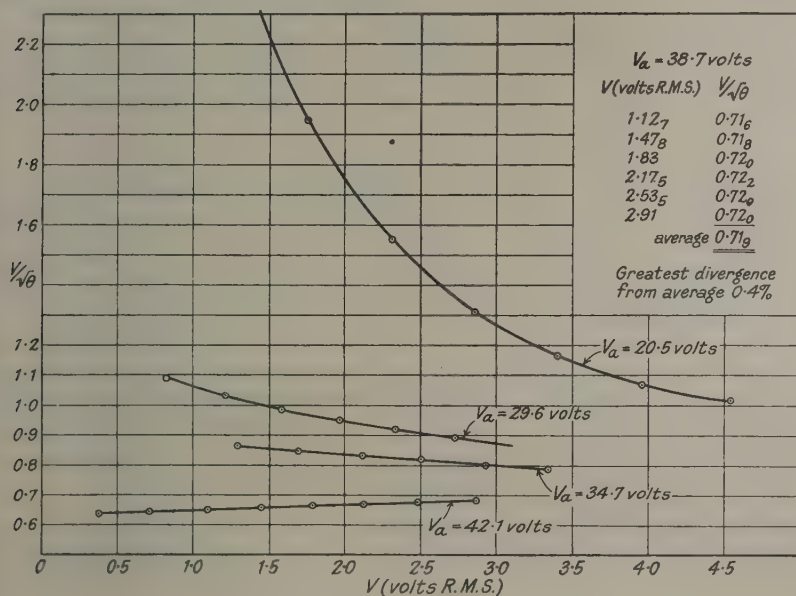


FIG. 3.

Several D.E.R. valves have been used from time to time, and little difficulty has been experienced in determining the adjustments necessary for compliance with the square-law within about 0.5 per cent.

From the results illustrated in Fig. 3 it is seen that for a given filament emission a grid bias and an anode voltage may be found for which the square root of the deflection on a good pivoted moving-coil galvanometer is proportional to the applied A.C. voltage within 0.4 per cent. The values of $V/\sqrt{\theta}$ given in the table for $V_a = 38.7$ are seen to rise and then fall slightly with increase of V , giving little hope that a better degree of constancy would be obtained for any neighbouring value of V_a .

* "Thermionic valves with dull-emitting filaments." M. Thompson and A. C. Bartlett, Journ. I.E.E. (April, 1924).

It is sufficient, however, for most measurements. This slight increase and decrease has been observed for all the valves so far examined.

It was subsequently decided to set the anode voltage at a definite value by means of the voltmeter shown in Fig. 4, and to adjust the grid-bias voltage by means

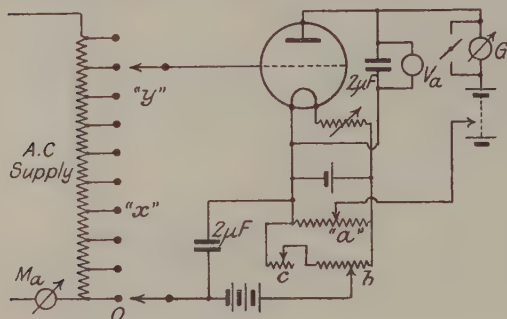


FIG. 4.—ILLUSTRATING THE ADJUSTMENTS NECESSARY TO RENDER $V/\sqrt{\theta}$ APPROXIMATELY CONSTANT.

"a" and "b" are 300 ohm potential dividers, "c" being a 30 ohm rheostat to give fine adjustment on the latter. *Ma* is a milliammeter used for calibration. *G*, M.C. galvanometer, with short-circuiting key. *V_a*, voltmeter.

of the resistances "b" and "c" till the instrument followed the square law as closely as possible. This was merely to avoid switching the voltmeter over to the grid battery.

The scheme finally devised for determining the necessary voltages for a given valve, and for adjusting the instrument for use at any time subsequently to calibration, is outlined below.

The anode voltage is first set at a definite value (15 volts in the present case), which gives a reasonably large galvanometer deflection when the grid is connected to the negative end of the filament. The filament rheostat is then adjusted to bring this deflection to a definite value. A small carbon pellet resistance was found to be definitely superior to the wire variety for this purpose, as it allows very fine continuous adjustment, and is not liable to oxidation.

The anode voltage is then set at the value at which it is intended to work (34 volts in the present case), and the voltmeter is connected to the calibration circuit. This may conveniently be a potential-dividing resistance carrying a known current and giving a range of voltages between about 0.5 and 3.0.

The next step is to determine the value of grid bias which will ensure the instrument following the square-law as nearly as possible. For this purpose two studs are selected on the potential divider giving galvanometer deflections of about 1/3 and 2/3 full-scale respectively. These have been marked "x" and "y" in Fig. 4.

In the present case the ratio $\left(\frac{\text{resistance } y_0}{\text{resistance } x_0}\right)^2$ was 3.87. If the voltmeter con-

forms with the square law, therefore, the ratio of the two galvanometer deflections will have this value. A few readings are then taken with different settings of "b" and "c," and the resulting values of the ratio θ_y/θ_x are plotted to a base of θ_0 , θ_0 being the deflection of the galvanometer corresponding to zero A.C. volts. The value of θ_0 corresponding to $\theta_y/\theta_x=3.87$ is read off from the graph, and the grid bias is always adjusted on "b" and "c" to give this value when the voltmeter is set up for use. In taking these readings the current through the calibration potential-divider is adjusted in each case to give approximately the same value for θ_y .

Briefly, the sequence of adjustments necessary before using the instrument are as follows :—

(a) Switch on the filament current and allow about five minutes for the valve temperatures to steady down.

(b) Set the anode voltage to a recognised voltage (say 15), and connect the grid to negative filament lead. Set the filament emission to a recognised value, as indicated by the galvanometer.

(c) Set the anode voltage to the recognised operating value (say 34), and adjust the grid bias to give a second recognised galvanometer deflection.

Once the two standardising values of galvanometer deflection have been determined, the operation of setting up the voltmeter for use is only a matter of a few minutes. With a valve such as one of the D.E.R. type there is a range of voltage adjustment on the potential dividers of two volts, any larger adjustments in grid or anode battery voltage being effected by adding or subtracting unit dry cells.

The results of the calibration tests carried out at different times on this voltmeter may be summarised as follows :—

The mean value of $V/\sqrt{\theta}$ was deduced from seven readings corresponding to applied potentials of from about 0.6 to 2.8 volts.

TEST No. 1, mean value 0.725. Greatest departure from the mean, 0.004.

TEST No. 2, mean value 0.719. Greatest departure, 0.006.

TEST No. 3, mean value 0.721. Greatest departure, 0.002.

The differences between the mean values in the three tests is not necessarily due to changes in the valve-voltmeter, as the milliammeter used in calibration was an air-enclosed thermo-junction.

(3) The usefulness of rapid indication in the indicating instrument used will not be overlooked by anyone making a large number of high frequency resistance measurements. A pivoted and critically damped moving-coil galvanometer of short period should be used.

DISCUSSION.

Dr. E. MALLETT congratulated the author on having made some fairly exact measurements bearing on the accuracy of valve voltmeters, which are of increasing importance at wireless as well as at telephonic frequencies. The previous results quoted by the author had suggested the liability of these instruments to startling inaccuracies, and it was reassuring to learn that such inaccuracies could be avoided.

Mr. T. I. JONES : I welcome the results of Mr. Sutton's measurements of valve capacities, and I am struck with the general agreement shown between these results at 650 kilocycles per second, and those for the normal cases quoted by Mr. Hartshorn and me as measured at 800 cycles per second. Mr. Sutton has quoted in his Paper some figures of ours, and has proceeded in contravention of our express warning to apply to them the analysis erroneously implied in Fig. 1. This figure deliberately assumes that each of the portions *A*, *B*, *C*, etc., of the electrodes successively removed could be represented by a simple direct capacity in parallel with the others and that they were directly additive, which would only obtain if each was completely shielded. The simplest system to represent the situation approximately would require two cross capacities to the portions of the electrodes still remaining in each individual case. Even when the direct capacity was free from loss the cross capacities would not be, as the lines of force invariably penetrated into dielectrics other than air. The measure of agreement obtained in the normal cases between the results at radio and at audio frequencies would indicate that the losses are not to be too closely associated with a shunt or an insulation resistance. Thus we see that the results of the analysis as given in Table I are misleading, and that the subsequent comments are uncalled for. I would point out that the figures quoted were obtained on a burnt out valve

which was broken open for the purpose. The P.F. is altogether higher than was found in the normal cases. Whether the surface of the glass pinch had been affected was not ascertained.

Dr. F. H. RAYNER referred to the table inset in Fig. 3, showing the variation of $V/\sqrt{\theta}$ with V , where V is the voltage to be measured and θ is the deflection observed. This variation appeared to be of the type which would result from the movement of the galvanometer coil in a field not strictly uniform.

The AUTHOR, in reply to the discussion, said that some of the valves tested had been broken open, while others had been intact. Possibly the effect of cross-capacity required fuller consideration than that which he had already given it. The galvanometer was a moving-coil Weston instrument; it had been calibrated with direct current, but the calibration curve did not confirm Dr. Rayner's suggestion.

Mr. R. P. FUGE (subsequent communication): I would like to ask Mr. Sutton if it is workable to put a very small condenser of the order of a fraction of a μ farad in series with the grid circuit of the valve in the voltmeter and so to reduce its capacity; and if the sensitivity of the voltmeter can be increased by amplifying by a second valve.

The AUTHOR (subsequent communication): In further reply to Mr. Jones, after giving further consideration to the effect of cross-capacities on the analysis suggested in Fig. 1, I am of the opinion that their neglect is permissible under the circumstances. If we consider, for instance, the cross-capacities between the electrodes and the leads in the pinch, to scale, it is obvious that the dielectric is almost entirely air. It seems most improbable that the small portion of the flux-path, which lies in the pinch itself, can account for such a high value of P.F. as 13 per cent.

The analysis was only intended to be a tentative one. The obvious assumptions made in replacing distributed capacities by localised condensers preclude any hope of precise deductions. It should represent the true conditions sufficiently accurately, however, to enable an estimate of the P.F. of the various portions to be made.

In reply to Mr. Fuge: The inclusion of a grid-condenser would be undesirable for two reasons. It would necessitate the use of a grid-leak; and any reduction in effective capacity would be accompanied by a corresponding reduction in sensitivity, the two condensers forming a potential divider. If previous amplifying valves are employed, the input impedance of the voltmeter depends on the circuit employed, owing to the "Miller" effect.

IV.—I. THE REFRACTION AND DISPERSION OF (1) AIR, (2) OXYGEN, (3) GASEOUS CHLOROFORM; II. NEW DETERMINATIONS OF THE GASEOUS REFRACTIVITIES OF (1) ACETONE, (2) METHYL ETHER, (3) ETHYL ETHER.

By H. LOWERY, *M.Sc., F.Inst.P.*, Head of the Physics Department, Technical College, Huddersfield.

Received July 8, 1927.

(Communicated by PROF. W. L. BRAGG.)

ABSTRACT.

I. The refractivities of air, oxygen and gaseous chloroform have been found for the green mercury line ($\lambda 5461$), and the dispersion studied over the range $\lambda 4830$ to $\lambda 6700$, with the following results :—

(1) Air :	$(\mu-1)_{n.t.p.} \times 10^7 = 2931(\lambda 5461)$
	$(\mu-1)_{n.t.p.} = \frac{4.8286 \times 10^{27}}{16774 \times 10^{27} - \nu^2}$
(2) Oxygen :	$(\mu-1)_{n.t.p.} \times 10^7 = 2711(\lambda 5461)$
	$(\mu-1)_{n.t.p.} = \frac{3.4343 \times 10^{27}}{12970 \times 10^{27} - \nu^2}$
(3) Gaseous chloroform :	$(\mu-1)_d \times 10^6 = 1412(\lambda 5461)$
	$(\mu-1)_d = \frac{14.622 \times 10^{27}}{10659 \times 10^{27} - \nu^2}$

where ν is the frequency of the light, $(\mu-1)_{n.t.p.}$ is the refractivity for N.T.P. conditions, and $(\mu-1)_d$ shows the refractivity by the same number of molecules of the gas as 1 cubic cm. of hydrogen contains at N.T.P.

II. The gaseous refractivities of acetone, methyl ether and ethyl ether for $\lambda 5461$ were found to be :—

(1) Acetone :	$(\mu-1)_d \times 10^6 = 1096,$
(2) Methyl Ether :	$(\mu-1)_d \times 10^7 = 8876,$
(3) Ethyl Ether :	$(\mu-1)_d \times 10^6 = 1509.$

I.

(1) AIR.

Introduction.

WHEN C. and M. Cuthbertson* published the results of their experiments on the refraction and dispersion of air in 1909 they reviewed the measurements of previous observers, and came to the surprising conclusion that "the accuracy of a determination is not a function of its date."

Meggers and Peters,† writing ten years later, also found occasion to draw attention to the "rather wide disagreements" between the observations up to that date, which, in addition to the results reviewed by C. and M. Cuthbertson, included work by Gruschke (1910), Siertsma (1913), Howell (1915), and Dickey (1917).

On examining the results of previous observers up to the present time (including

* Proc. Roy. Soc., A, Vol. 83, p. 151 (1909).

† Bulletin, Bureau of Standards, Washington, XIV., No. 4, p. 697 (1919).

the work of Meggers and Peters, Traub* and Stoll†), one still finds call to remark upon the considerable differences among the various determinations.

No apology is, therefore, necessary for publishing some additional results on the refraction and dispersion of air (stated for precisely known conditions), although the writer is fully conscious of the limited range of his dispersion curve and of the need for extending it as far as possible into the ultra-violet and infra-red regions, as was attempted, for example, by Meggers and Peters.

Experimental.

The refractive index of dry air (not deprived of carbon dioxide) for the green mercury line ($\lambda 5461$) was determined by means of a Jamin interferometer with the following results:—

$$(\mu-1) \times 10^7 \quad . \quad . \quad . \quad 2926, 2929, 2937, 2926, 2934, 2932.$$

$$\text{Mean : } 2931 \text{ (}\lambda 5461\text{)}.$$

Each of these values expresses the refractive index reduced to N.T.P. conditions by the formula

$$\mu-1 = \frac{n\lambda}{l} \cdot \frac{760}{p} \cdot \frac{273+t}{273},$$

μ being the refractive index, n the number of interference bands counted, λ the wavelength of the light, l the length of the refraction tubes, p the difference in pressure between the tubes and $t^\circ\text{C}$. the temperature of the experiment. About 350 bands were counted, reading to one-tenth of a band.

The dispersion was measured in relation to the value of the refractive index for $\lambda 5461$ by the method of C. and M. Cuthbertson (loc. cit., p. 152). (My thanks are due to Mr. Cuthbertson for loaning me his original compensator.)

The experimental results in the second column of Table I lead to the Sellmeier expression:—

$$\mu-1 = \frac{4.8286 \times 10^{27}}{16774 \times 10^{27} - \nu^2}$$

(ν being the frequency of the light), from which the values of the third column have been calculated by the method of least squares.

TABLE I.—*Dispersion of Air.*

λ in Å.U.	$(\mu-1) \times 10^7$.		Difference.
	Observed.	Calculated.	
Li 6708	2913, 4	2913, 4	0
H α 6563	2914, 5	2914, 9	+4
Cd 6438	2916, 1	2916, 4	+3
Hg 5790	2925, 0	2925, 4	+4
Hg 5461	2931, 1	2931, 4	+3
Ag 5209	2936, 8	2936, 7	—1
Cd 5086	2940, 0	2939, 6	—4
H β 4861	2946, 1	2945, 5	—6
Cd 4800	2947, 9	2947, 3	—6

* Ann. d. Phys., (4), LXXI., p. 533 (1920).

† Ann. d. Phys., (4), LXXIX., p. 81 (1922).

Previous Work.

This has been so thoroughly reviewed by C. and M. Cuthbertson and Meggers and Peters that it only remains to state the following most recent values of the refractive index for $\lambda 5461$ and N.T.P. conditions for comparison with the above :—

1.0002925 (Meggers and Peters, 1919).

1.0002933 (Traub, 1920).

1.0002927 (Stoll, 1922).

C. and M. Cuthbertson's values of the constants in the Sellmeier formula (single term) are : $C = 4.6463 \times 10^{27}$ and $\nu_0^2 = 16125 \times 10^{27}$.

(2) OXYGEN.

Experimental.

The oxygen for the present work was prepared by heating pure recrystallised potassium permanganate. It was then washed by passage through a strong solution of caustic potash to remove traces of carbon dioxide, and dried over calcium chloride and phosphorus pentoxide.

Six experiments to determine the refractive index for $\lambda 5461$ (Hg) and N.T.P. conditions yielded the following results :—

$(\mu - 1) \times 10^7$ 2708, 2720, 2706, 2709, 2716, 2705.

Mean 2711 ($\lambda 5461$).

The results of the dispersion measurements are shown in Table II, together with the values calculated from the Sellmeier expression :—

$$\mu - 1 = \frac{3.4343 \times 10^{27}}{12970 \times 10^{27} - \nu^2}.$$

TABLE II.—Dispersion of Oxygen.

λ in Å.U.	$(\mu - 1) \times 10^7$		Difference.
	Observed.	Calculated.	
Li 6708	2688, 9	2689, 4	+5
Cd 6438	2693, 0	2693, 0	0
Hg 5770	2704, 8	2704, 2	—6
Hg 5461	2711, 2	2711, 0	—2
Ag 5209	2717, 4	2717, 4	0
Cd 5086	2720, 6	2720, 9	+3
Cd 4800	2729, 6	2730, 1	+5

Previous work.

The following values of the refractivity for $\lambda 5461$ serve for comparisons between the results of previous workers and the present :—

	$(\mu - 1) \times 10^7$
Rentschler (1908)	2725
Ahrberg (1909)	2706
C. and M. Cuthbertson (1909)	2717
Koch (1909)	2704
Howell (1915)	2728
Stoll (1922)	2718

[N.B.—The values of Howell and Stoll have been read from their dispersion curves.]

(3) CHLOROFORM.

Previous work.

Mascart,* working at a temperature "about 12°C.," found the refractivity of gaseous chloroform, relative to that of air, to be 4.98 for sodium light. Lorenz† found the refractive indices to be 1.001442 and 1.001435 for $\lambda 5893$ and $\lambda 6708$ respectively. These are the only recorded determinations.

Experimental.

The experimental details are exactly the same as those in the work on gaseous carbon disulphide already described.‡ The greatest difference in pressure between the two refraction tubes was 20 cms. of mercury and about 250 interference bands were counted, reading to one-tenth of a band.

Pure liquid chloroform, supplied by Messrs. British Drug Houses, Ltd., was used in the experiments.

(a) *Refractive Index of Gaseous Chloroform.*—The refractive index of gaseous chloroform for the green mercury line ($\lambda 5461$) was determined in 10 experiments with the following results:—

$(\mu-1) \times 10^6$. . . 1409, 1418, 1408, 1411, 1419, 1410, 1408, 1409, 1416, 1410.

Mean: 1412 ($\lambda 5461$).

Each of these results was calculated from the expression

$$\mu-1 = \frac{n\lambda}{l} \cdot \left(\frac{\text{Standard density}}{\text{Density observed in experiment}} \right).$$

Assuming C=12.000, H=1.008, Cl=35.457, and the density of hydrogen = 0.08985 grams per litre, the standard density of gaseous chloroform is 5.321 grams per litre.

The mean experimental density obtained was 5.445 grams per litre. Dumas and Regnault found the density of gaseous chloroform, relative to that of air, to be 4.199 and 4.230 respectively. When reduced, these values become 5.429 and 5.469 grams per litre.

(b) *Dispersion of Gaseous Chloroform.*—The results of the dispersion measurements, together with the values of $(\mu-1) \times 10^6$ calculated from the expression

$$\mu-1 = \frac{14.622 \times 10^{27}}{10659 \times 10^{27} - \nu^2},$$

are shown in Table III.

TABLE III.—*Dispersion of Gaseous Chloroform.*

λ in Å.U.	$(\mu-1) \times 10^6$		Difference.
	Observed.	Calculated.	
Li 6708	1397, 8	1398, 0	+2
Cd 6438	1400, 2	1400, 3	+1
Li 6104	1403, 7	1403, 6	-1
Hg 5770	1407, 8	1407, 5	-3
Hg 5461	1412, 0	1411, 8	-2
Cd 5086	1418, 1	1418, 1	0
Cd 4800	1423, 5	1424, 0	+5

* Comptes Rendus, LXXXVI, p. 1182 (1878).

† Wied. Ann., XI, p. 70 (1880).

‡ Proc. Phys. Soc., Vol. 38, p. 470 (1926).

(c) *The Refractive Index of Gaseous Chloroform for Sodium Light (λ5893) under N.T.P. Conditions.*—This value may readily be obtained from the above data, thus the Sellmeier formula gives the value of $(\mu-1)$ for λ5893 to be 0.001406. On altering this in the ratio,

$$\frac{\text{Experimental density at N.T.P. } 5.445}{\text{Standard density } 5.321}$$

the refractive index becomes 1.001439 which is the required result.

The corresponding values of Mascart and Lorenz are 1.001455 and 1.001442 respectively.

II.

(1) ACETONE.

Previous Work.

Mascart* found the refractivity of gaseous acetone, relative to that of air, to be 3.74 for λ5893, that is, assuming the refractive index of air for λ5893 to be 1.0002923, his determination of the refractive index of acetone becomes 1.001093. Prytz† obtained the values 1.001085 and 1.001079 for λ5893 and λ6708 respectively. No other determinations are recorded.

Experimental.

Six experiments were performed in order to determine the refractive index in relation to the density for the green mercury line (λ5461) with the following results :—

$$(\mu-1) \times 10^6 \quad . \quad . \quad 1099, 1092, 1101, 1099, 1094, 1092.$$

$$\text{Mean : } 1096 (\lambda 5461).$$

These results were calculated as in the case of chloroform above. About 300 interference bands were counted in each experiment. The standard density of gaseous acetone is 2.587 grams per litre. The mean experimental density obtained was 2.594 grams per litre.

(2) METHYL ETHER.

Previous Work.

The only recorded determination of the refractivity of this gas is that of Mascart (loc. cit.), who obtained the value 3.03 relative to that of air, leading to 1.000885 as the refractive index for λ 5893.

Experimental.

The methyl ether employed was prepared and purified by the method of Erlenmeyer and Kreichbaumer.‡ The refractive index was found in relation to the density from six experiments (about 400 bands being counted in each experiment), thus :—

$$(\mu-1) \times 10^7 \quad . \quad . \quad 8870, 8879, 8882, 8871, 8875, 8878.$$

$$\text{Mean : } 8876 (\lambda 5461).$$

* Loc. cit.

† Wied. Ann., XI, p. 104 (1880).

‡ Berichte, VII., p. 699 (1874).

The theoretical density of methyl ether is 2.052 grams per litre. The mean experimental density is 2.099 grams per litre.

(3) ETHYL ETHER.

Previous Work.

The following values are recorded by Dufet, *Recueil des Données Numériques*, Vol. I (1900). No other determinations appear to have been made.

λ	$(\mu-1) \times 10^6$ at N.T.P.	Observer.
White.	1554	Arago.*
White.	1523	Dulong.†
D	1534	Mascart.‡
Li.	1517	Lorenz.§
D	1524	"

Experimental.

The results of six experiments to determine the refractive index in relation to the density yielded the following results:—

$$(\mu-1) \times 10^6 \quad . \quad . \quad . \quad 1506, 1513, 1505, 1507, 1515, 1506.$$

$$\text{Mean : } 1509 (15461).$$

The theoretical density of ethyl ether is 3.298 grams per litre. The mean density found in the present experiments is 3.342 grams per litre.

The author's thanks are again due to Prof. W. L. Bragg, F.R.S., and Mr. C. Cuthbertson, F.R.S., for the loan of apparatus and much advice.

* *Mém. Sci.*, II., p. 711 (1859).

† *Ann. de Ch. et Phys.* (2 Sér.), XXXI., p. 154 (1826).

‡ *Loc. cit.*

§ *Loc. cit.*

V.—THEORY OF THE ELASTIC PIANOFORTE HAMMER.

By PANCHANON DAS, M.Sc.

(Communicated by Prof. C. V. RAMAN, F.R.S.)

ABSTRACT.

In previous Papers exact expressions have been obtained for the vibrations set up in a string by the impact of an inelastic and also of an elastic hammer. By making various approximations, the influence of which is examined in the Paper, formulæ are obtained from which it is easy to deduce the practical effect of the elasticity and velocity of the hammer, and of the position of its point of impact on the string.

INTRODUCTION.

SINCE Professor Bryan* first brought the subject before the Physical Society, numerous Papers have appeared on the theory of the pianoforte. Professor Raman† and Dr. Banerji investigated the case of a string excited by an inelastic hammer by considering the motion during the impact as equivalent to that of a loaded string. The present writer‡ has discussed the subject in a series of theoretical Papers, giving an exact mathematical solution of the problem, first, of an inelastic hammer, and, later, of an elastic hammer, whose compression is proportional to pressure. On the experimental side, Datta§ has investigated the harmonics of a struck string, while George,|| and, later, George and Beckett,¶ have published a series of Papers on the dynamics of the hammer and the energy of the struck string. Papers have also appeared by Ghosh,** and Bhargava and Ghosh.††

It is proposed in this Paper to discuss a topic which has not yet been fully cleared up, namely, the purpose served by the elasticity of the hammer. In order to bring out the contrast between the rigid and the elastic hammers, it will be found convenient to consider a theory of both in simplified forms.

I. SIMPLIFIED THEORY OF INELASTIC HAMMER.

Kaufmann,‡‡ as is well known, assumed as an approximation that the shorter section of the string remains straight during the impact of the hammer, and in terms of the following symbols :—

a = shorter section of string	c = wave-velocity of string
l = whole length of string	m = mass of hammer
ρ = line-density of string	V = initial velocity of hammer
T = tension of string	P = pressure of impact of hammer

* G. H. Bryan, Proc. Phys. Soc., Vol. 25 (1912-13).

† C. V. Raman and B. Banerji, Proc. Roy. Soc., Vol. 97, p. 108 (1920).

‡ P. Das, Proc. Ind. Assoc. Cult. Sc., Vol. 7, Pt. 1; Vol. 9, Pt. 4; Vol. 10, Pt. 1.

These will be referred to as No. A, B, C.

§ Proc. Ind. Assoc. Cult. Sc., Vol. 8, p. 107.

|| Proc. Roy. Soc., 108, p. 284 (1925). Phil. Mag. 49, p. 97, and 50, p. 491 (1925).

¶ Proc. Roy. Soc., 114, p. 111 (1927).

** Phil. Mag., p. 875 (1926).

†† Phil. Mag., 49, p. 121, and 47, p. 1141.

‡‡ Ann. der Phys., Bd. 54 (1895).

obtained an expression for the pressure of the hammer in the following form :—

$$P = \frac{V}{c\eta} \cdot e^{\xi ct} \sin(\eta ct + \theta), \text{ where } \xi = -\frac{\rho}{2m},$$

$$\eta = \sqrt{\frac{\rho}{ma} - \frac{\rho^2}{4m^2}}$$

$$\text{and } \theta = 2 \tan^{-1}\left(\frac{\eta}{\xi}\right).$$

In reality, however, this shorter section of the string is also set in vibration during the impact. This gives rise to an additional periodic term in the expression for pressure, the form of which is readily derived from the nature of the vibrations set up. Graphically, it is a periodic zig-zag, which can be expressed by a Fourier series. Thus the complete expression for the pressure of the hammer is

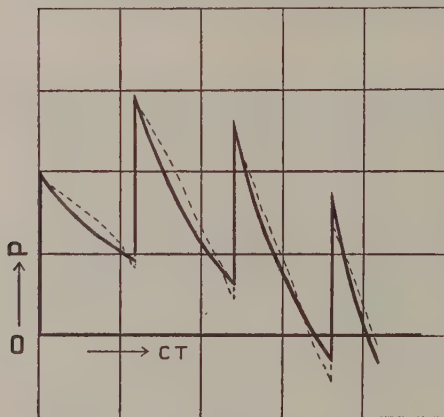


FIG. 1

$$P = \frac{V}{c\eta} e^{\xi ct} \sin(\eta ct + \theta) + \frac{\rho V c}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi ct}{a} \quad (1)$$

The curve of P against ct is drawn in Fig. 1, the dotted curve being derived from (1) and the continuous one from Das's exact formula.*

It will be noted that formula (1) approximates very closely to the results of exact solution.

II. ELASTIC HAMMER.

In this case, the natural vibrations of the section a of the string will later be shown to be of small amplitude. We may therefore at first follow Kaufmann and assume it to remain straight during the impact. Later, we shall also take into account the vibrations of the shorter section a .

Using the following notation :—

- y_0 = displacement of string at $x=0$
- y_1 = " " " " for $x < 0$
- y_2 = " " " " " $x > 0$
- ξ = " " " hammer
- u = compression " "
- E = elastic constant of hammer

* Paper No. 1 (l.c.).

we are led to put down :—

$$\begin{cases} y_1=f(x+ct) \\ y_2=\frac{a-x}{a}y_0 \\ y_0=f(ct) \end{cases}$$

The equations of motion are easily seen to be :—

$$m_0 \frac{d^2 \xi}{dt^2} = T \left(-\frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial x} \right)_{x=0} = -Eu \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the term Eu being equal to the pressure P and m_0 being $m + \rho a/3$ (Kaufmann's correction).

Denoting by accents the differentiations with respect to time, we get

$$u = \frac{T}{E} \left(\frac{1}{a} y_0 + y_0' \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, since the displacement ξ of the hammer equals the sum of the displacement y_0 of the string and the compression u , we have

$$\xi = y_0 + u \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus, we have from (3) and (4) :—

$$m_0 (y_0'' + u'') = -Eu \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting (3) in (5), we get :—*

$$\frac{m_0 T}{Ec} y_0''' + \left(1 + \frac{T}{aE} \right) y_0'' + \frac{T}{c} y_0' + \frac{T}{a} y_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The complete integral of (6) is

$$y_0 = A e^{q\xi} + B e^{\xi\alpha} \sin(\eta ct + \theta) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where A, B, θ are constants of integration and $q, \xi \pm i\eta$ are the roots of

$$x^3 + \left(\frac{E}{T} + \frac{1}{a} + \frac{\rho a}{3m_0} \right) x^2 + \frac{\rho}{m_0} \cdot \frac{E}{T} \left(x + \frac{1}{a} \right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From the initial conditions, $y_0=0, y_0'=0, \xi'=V$, we get :—

$$\left. \begin{aligned} A &= \frac{V}{c} \cdot \frac{E}{T} \cdot \frac{1}{\eta^2 + (q - \xi)^2} \\ B &= \frac{V}{c} \cdot \frac{E}{T\eta} \cdot \frac{1}{\sqrt{\eta^2 + (q - \xi)^2}} \\ \theta &= \tan^{-1} \frac{\eta}{q - \xi} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

* Bhargava and Ghosh (l.c.) derived a formula in which they omitted the term y_0''' occurring in (6). Thus their value of pressure of impact at $t=0$ is finite, which cannot be true, showing that their formula is invalid.

We note that the dimension of the roots of (8) is L^{-1} , and both ρ/m_0 and E/T also have the same dimension, and therefore these are comparable. If ρ/m_0 is small compared with the largest root, a further simplification arises. The equation (8) has, then, the following approximate roots :—

$$\left. \begin{aligned} q &= -\left(\frac{E}{T} + \frac{1}{a}\right) \text{ and } \xi \pm i\eta \\ \xi &= -\frac{aE}{aE+T} \cdot \frac{\rho}{2m_0} \\ \eta &= \sqrt{-\frac{2\xi}{a} - \xi^2} \end{aligned} \right\} \dots \dots \dots (10)$$

where
and

The expression for the pressure of impact is then given by

$$P = \frac{V}{c} \cdot \frac{E^2}{Eq^2} (e^{\xi ct} \cos \eta ct - e^{\eta ct}) - \frac{V}{c} \cdot \frac{E}{qa\eta} \cdot e^{\xi ct} \sin \eta ct \dots \dots (11)$$

which may also be written in the form

$$P = A_1 e^{\xi ct} + B_1 e^{\xi ct} \sin(\eta ct + \theta_1) \dots \dots \dots (12)$$

III. COMPARISON WITH EXACT THEORY.

Taking the particular case of a string and hammer system for which $\rho/m_0 = 0.01$ and $E/T = 0.4$, the curve (continuous one) of P against ct has been drawn in

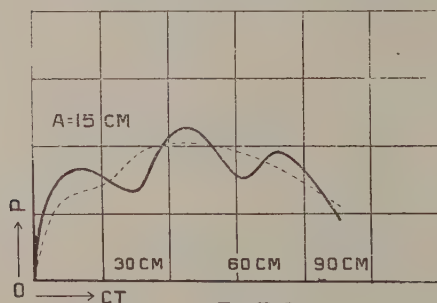


FIG 2.A.

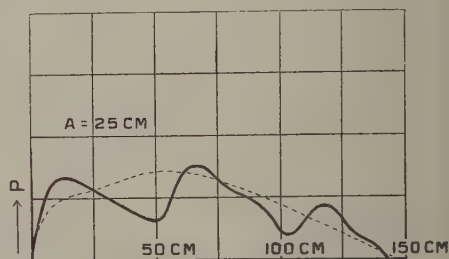


FIG. 2.B.

Fig. 2 (A, B), with the help of Das's exact formula* for $a=15$ cm. and $a=25$ cm. respectively.

The dotted curves in the same figures are derived from the approximate formula (11).

It will be observed that the general trend of the exact curve is well indicated by the approximate curve, except that the former is characterised by a superimposed wave-curve due to the vibrations of a .

On comparing Fig. 1 with Fig. 2, we find that as the rigid hammer is replaced by an elastic one, the discontinuous periodic rises in pressure lose their sharp angularities and give place to well-rounded humps. In Fig. 3 the pressure-time curve for

* Paper No. C, Art. I.

only one vibration of a has been drawn for different values of E/T , other things remaining the same. It will be seen that as the value of E is diminished these humps become less and less prominent.

IV. HARMONICS OF STRUCK STRING.

The s^{th} normal co-ordinate φ_s in the vibration of the string resulting from the impact is given by

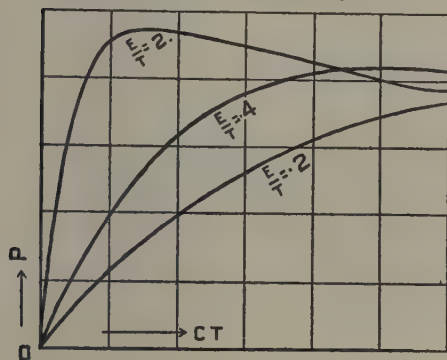


FIG. 3.

$$\varphi_s = \frac{2}{n \rho l} \int_0^t \sin n(t-t') \Phi_s dt',$$

where $n = s\pi c/l$, and Φ_s is the generalised component of force, here given by

$$\Phi_s = P \sin \frac{s\pi a}{l}$$

P being the expression (12). Now, the duration of impact is approximately given by

$$t_1 = \frac{1}{\eta c} \tan^{-1} \frac{E}{T} \cdot \frac{a\eta}{q} \dots \dots \dots (13)$$

Hence the time of action of the force Φ_s is given by (13). In order to simplify the integration involved in φ_s we take advantage of the fact that ξct is very small and qct_1 is a large negative quantity. So we at once put $e^{\xi t} = 1$, and after integration put $e^{\xi t_1} = 0$, and get

$$\begin{aligned} \varphi_s = & \frac{2}{n \rho l} \cdot \frac{V}{c} \sin \frac{s\pi a}{l} \cdot A_1 \left[-\frac{qc \sin nt + n \cos nt}{q^2 c^2 + n^2} \right] \\ & + \frac{2}{n \rho l} \cdot \frac{V}{c} \sin \frac{s\pi a}{l} \cdot B_1 \left[\frac{\eta (\cos nt_1 - \cos \theta_1)}{n^2 - \eta^2 c^2} \sin nt \right. \\ & \left. - \frac{\eta \sin nt_1 + n \sin \theta_1}{n^2 - \eta^2 c^2} \cos nt \right] \dots \dots \dots (14) \\ = & a_s \sin nt + C_s \cos nt, \text{ say.} \end{aligned}$$

Then the amplitude of the harmonic of order S is given by

$$\sqrt{a_s^2 + b_s^2}$$

As experimental data on the elastic hammer are rather meagre, we content ourselves with a few general observations only on the formula (14).

It is obvious from the term having $n^2 - \eta^2 c^2$ as denominator that, if n or $s\pi c/l$ happens to be nearly equal to ηc , the corresponding harmonic will have a

large amplitude. Since the value of $\pi/\eta c$ usually lies near half the period of vibration of the string, viz. l/c , it is apparent that the value of s satisfying this condition is unity. Hence it is generally the fundamental that has the largest amplitude.

In order to understand how the elasticity of the hammer modifies the tone-quality, it is also necessary to consider the vibrations of the section a of the string. The pressure-component arising out of these vibrations may be expressed as a Fourier series :—

$$\sum_1^{\infty} A_r \sin \frac{\pi r c t}{a} \dots \dots \dots (15)$$

From what has been remarked before, it is obvious that when we pass on to the limiting case of the rigid hammer by putting $E = \infty$ the series (15) should become identical with the additive series in (1). For finite values of E , the softer the hammer the smaller the amplitudes of these vibrations (*see* Art. III), hence the smaller are the values of A_r .

Now, the contribution to the amplitude of the harmonic of the string, say, of order s , by these terms is seen to be

$$\frac{2}{n \rho l} \sin \frac{s \pi a}{l} \int_0^t \sin n(t-t') \sum_1^{\infty} A_r \sin \frac{\pi r c t'}{a} dt'.$$

If we retain only the most significant terms, it reduces to

$$\begin{aligned} & \frac{1}{n \rho l} \sin \frac{s \pi a}{l} \sin n t \sum A_r \cdot \frac{1}{n - \frac{\pi r c}{a}} \cdot \left\{ \cos \left(n - \frac{\pi r c}{a} \right) t_1 - 1 \right\} \\ & - \frac{1}{n \rho l} \sin \frac{s \pi a}{l} \cos n t \sum A_r \cdot \frac{1}{n - \frac{\pi r c}{a}} \cdot \sin \left(n - \frac{\pi r c}{a} \right) t_1 \dots \dots \dots (16) \end{aligned}$$

It is obvious from the denominator $n - \pi r c/a$ that if (barring absolute equality) n happens to be nearly equal to $\pi r c/a$, then the contribution to the corresponding harmonic is likely to be appreciable. Thus we find that if a harmonic of the string has nearly the same period as a harmonic of the section a , the former has considerable amplitude. If the elasticity of the hammer is varied under these circumstances, then, from what has been said about the coefficients A_r , it is obvious that the reinforced harmonics in question will be most pronounced when the hammer is rigid (values of A_r being then maximum) and will be weakened as the hammer is replaced by softer ones. This explains why a hard hammer imparts a piercing quality to the general tone of the vibrating string.

In order to study the effect of altering the length a , let us, first, consider the following special values, $a = l/K$, where $K = s, s/2, s/3, s/4$, etc. One sees from the factor $\sin (s \pi c/l)$ that the expression (16) vanishes for these values of a . But if a is slightly different from these values the expression (18) is fairly large for the values of r , which make n nearly equal to $\pi r c/a$. Thus we see that as a is varied

the overtones pass through a series of maxima* separated by small or minimum values.

Lastly, we examine the relation between the initial velocity of the hammer and the intensities of overtones. From the factor V occurring in (14) it is obvious that the *relative* intensities are unaffected by a change in the value of V . But it appears that, if the hammer undergoes a large compression, the corresponding pressure develops much more rapidly than is to be expected from Hooke's law, which is the basis of our theory. This is much the same as if the elasticity coefficient E does not remain constant, but rather increases with compression. Thus, if the hammer has a large initial velocity and hence undergoes a large compression, its effective hardness increases. Then from what has been said about the hard hammer one concludes that if the string is struck very hard the tone loses its softness and acquires a metallic ring.

My best thanks are due to Prof. C. V. Raman for his inspiring suggestions and continual guidance in this present work.

* See Datta (l.c.).

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